Hats Brainteaser

Ed Felten

The Puzzle

There are K players seated around a table, on which there is a closed box containing colored hats. The hats come in K colors, and there are K hats of each color in the box. (The players are given a list of the hat colors that are in the box.) The players close their eyes, and a referee puts a hat on each player's head, with each hat chosen randomly from the box.

Now the players open their eyes. Each player can see the colors of the other players' hats, but cannot see their own hat. The players cannot communicate in any way.

Each player now seals in an envelope a guess as to the color of their own hat. They can't see each other's guesses, and they still can't communicate.

At the end, the envelopes are unsealed and each player's guess is compared to the color of their own hat. If at least one player guessed right, they all win as a team. But if they all guessed wrong, then they all lose as a team.

The puzzle is this: *Can you think of a strategy the players can agree on in advance that is guaranteed to win every time?*

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The Solution

To solve the hats puzzle, it helps to think of the hats as having numbers on them. There are K colors, and we can assign a number to each color, so the K colors get the numbers 0, 1, 2, ... , up to K-1. We might say that orange hats get 0, black hats get 1, and so on. It doesn't matter which number goes with which color, as long the players all agree on how the numbers are assigned.

Now the players coordinate their guesses as follows. The first player, Alice, assumes that the sum of everyone's hat value will be exactly a multiple of K. There is only one value for Alice's hat that will make her assumption correct, so she guesses that her hat has that value. The second player, Bob, assumes that the sum of hat values, plus one, is exactly a multiple of K; and he guesses the unique value that would make that assumption correct. The third player assumes that the sum of hat value, plus two, is a multiple of K, and guesses accordingly, and so on, with the last player assuming that the sum, plus K-1, is a multiple of K.

Following this scheme guarantees that at least one player will guess correctly. If a player's assumption is correct, that player will guess correctly. What is a little harder to see, but turns out to be true, is that exactly one player's assumption will be correct. That's true because the sum of all hat values must be either a multiple of K, or one less than a multiple of K, or …, or K-1 less than a multiple of K. We don't know which player's assumption will correct, but we know that exactly one will be correct, and we know that that player will correctly guess their own hat color. It follows that this strategy always wins.

If this is confusing, consider the case where K=2. Let's call the two players Alice and Bob, and let's assume the hat colors are orange and black. We'll assign the number zero to orange, and one to black. Now each player gets a randomly chosen hat, which might as well have been picked by flipping a coin. Alice guesses that the two hats add up to a multiple of two, which can only happen if both hats are orange (0+0) or both hats are black (1+1), so Alice guesses that her hat is the same color as Bob's. Bob, on the other hand, assumes that the sum of the two hat colors, plus one, is a multiple of two, which can only happen if one hat is orange and the other is black (0+1, or 1+0), so Bob guesses that his hat is the opposite color to Alice's. One of the two assumptions must be right – the hats must be either the same color or different colors---so somebody will guess right. And now you can see, I hope, the link between the two puzzles I posed.

(If you can't get enough of these puzzles, here's another one: How can you make a fair random choice among K alternatives, with each alternative being chosen with probability exactly 1/K, using only honest dice in the shapes of Platonic solids, that is, having 4, 6, 8, 12, or 20 sides? Your method should work for all positive values of K.)